Noname manuscript No.
(will be inserted by the editor)

A permuted factors approach for the linearization of polynomial matrices

S. Vologiannidis · E.N. Antoniou

Received: date / Accepted: date

Abstract In [1] and [2] a new family of companion forms associated to a regular polynomial matrix $T(s)$ has been presented, using products of permutations of n elementary matrices, generalizing similar results presented in [3] where the scalar case was considered. In this paper, extending this "permuted factors" approach, we present a broader family of companion like linearizations, using products of up to $n(n-1)/2$ elementary matrices, where n is the degree of the polynomial matrix. Under given conditions, the proposed linearizations can be shown to consist of block elements where the coefficients of the polynomial matrix appear intact. Additionaly we provide a criterion for those linearizations to be block symmetric. We also illustrate several new block symmetric linearizations strictly equivalent to the original polynomial matrix $T(s)$ where in some of them, the constraint of nonsingularity of the constant term and the coefficient of maximum degree is not a prerequisite.

Keywords polynomial matrices · linearizations · realizations · companion matrix · linear systems

1 Introduction

Polynomial matrices arise in the study of several problems in the analysis and synthesis of linear multivariable systems (see for instance [13]). A very common form of representation of dynamical linear systems (see [12]) is the so called auto regressive (AR) representation

$$\Sigma : T(\rho)\xi = 0, \tag{1}$$

S. Vologiannidis
Technological Educational Institute of Serres, Department of Information and Communication Sciences, 62124 - Serres, Greece
Tel.: +302310997909
E-mail: svol@teiser.gr

E. Antoniou
Technological Educational Institute of Thessaloniki, Department of Sciences, 54101 - Thessaloniki, Greece
Tel.: +302310791429
E-mail: eantonio@gen.teithe.gr

where is either the differential or the forward difference operator, $T(\rho)$ is a polynomial matrix and ξ is the vector valued pseudostate. The properties of the dynamical system (1) are closely related to the structural invariants of the polynomial matrix

$$T(s) = T_n s^n + T_{n-1} s^{n-1} + ... + T_0, \tag{2}$$

where $T_i \in \mathbb{C}^{p\times p}$. A polynomial matrix $T(s)$ is said to be *regular* iff $\det T(s) \neq 0$ for almost every $s \in \mathbb{C}$. In practical applications, such as the eigenvalue problem of higher order systems, it is preferable to linearize the polynomial matrix $T(s)$ into an equivalent matrix pencil $L(s) = sL_1 - L_0$ where $L_i \in \mathbb{C}^{np\times np}$. Strictly speaking, linearizations share the same finite and infinite divisor structure with the original polynomial matrix. Several linearizations exist in the literature ([6], [11], [12], [5], [1], [2], [10], [8], [9]), the most common ones being linearizations using companion matrices and especially the first and second companion forms. Companion linearizations are linearizations where L_i are directly composed as block matrices using the coefficients T_i. For instance, the first companion linearization is $P(s) = sP_1 - P_0$ where

$$P_1 = \begin{bmatrix} I_p & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_n \end{bmatrix}, \; P_0 = \begin{bmatrix} 0 & I_p & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_p \\ -T_0 & -T_1 & \cdots & -T_{n-1} \end{bmatrix} \tag{3}$$

and the second one $\hat{P}(s) = sP_1 - \hat{P}_0$ where

$$\hat{P}_0 = \begin{bmatrix} 0 & \cdots & 0 & -T_0 \\ I_p & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & -T_{n-2} \\ 0 & \cdots & I_p & -T_{n-1} \end{bmatrix}. \tag{4}$$

Several authors have proposed other linearizations of a polynomial matrix $T(s)$. In [11], [6] two vector spaces of linearizations have been proposed and a convenient method for constructing them using shifted sums was presented (*additive construction approach*). The intersection of those vector spaces was proved to contain block symmetric linearizations. In [1] and [2], a family of linearizations was introduced using products of permutations of n elementary matrices (*permuted factors approach*), some of which proved to be also block symmetric. Notably, this approach has been shown to be extensible in the case of multivariate polynomial matrices in [7].The two approaches described so far provide different linearizations, but in both cases, crucial role is played by the first and second companion linearizations. A third approach for the construction of block symmetric linearizations appeared in [8], [9] and [10] using products of the first or second companion linearizations with a suitable block symmetric matrix (*multiplicative approach*).

In this paper we generalize the "permuted factors" approach using products of more than n elementary matrices, thus constructing new companion like linearizations. Note that from [1] and [2], the elementary matrices involved are

$$A_k = \begin{bmatrix} I_{p(k-1)} & 0 & \cdots \\ 0 & C_k & \ddots \\ \vdots & \ddots & I_{p(n-k-1)} \end{bmatrix}, \; k = 1, 2, \ldots, n-1, \tag{5}$$

$$C_k = \begin{bmatrix} 0 & I_p \\ I_p & -T_k \end{bmatrix} \tag{6}$$

and

$$A_0 = diag\{-T_0, I_{p(n-1)}\}. \tag{7}$$

Also $A_i, i = 1, ..., n-1$ are always nonsingular and A_0 is nonsingular if and only if T_0 is respectively nonsingular. A central role in this process is played by the newly introduced notion of *operation free* products of elementary matrices. Operation free products are essentially products resulting in block matrices containing only trivial blocks such as 0 or I_p and T_i. By avoiding operations between the coefficients T_i, it is guaranteed that the numerical data of the original problem are not perturbed. This is the case in the two standard companion linearizations, the ones in [1], [2] and also in those in [8], [9] and [10]. Operation free products can be viewed as a special case of intrinsic products in [4]. Characterizations of operation free products are given both in terms of adjacency and the existence of certain types of standard forms.

The rest of the paper is outlined as follows. In section 2, we study the family of matrices that arise using products of at most $n(n-1)/2$ elementary matrices A_i and give necessary and sufficient conditions for those products to be operation free. In section 3, we introduce a new family of companion like linearizations, whose matrices are operation free products. This family is proved to include the linearizations of [1], [2] but also the ones produced using the multiplicative "approach" in [8], [9] and [10]. Moreover, some of the linearizations of a particularly interesting block symmetric structure are pointed out, where in some of them, the constraint of nonsingularity of the constant term and the coefficient of maximum degree is not a prerequisite.

2 Operation free products of elementary matrices

In order to study the products of elementary matrices A_i, we will need the following introductory definitions and results.

Definition 1 Let $\mathcal{I} = (i_1, i_2, \ldots, i_m)$ be an ordered tuple containing indices from $\{0, 1, 2, \ldots, n-1\}$. Then $A_{\mathcal{I}} := A_{i_1} A_{i_2} \cdots A_{i_m}$.

Lemma 1 *[3],[1] Let $i, j \in \{0, 1, 2, \ldots, n-1\}$. Then $A_i A_j = A_j A_i$ if and only if $|i-j| \neq 1$.*

Definition 2 Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two tuples. $\mathcal{I}_1$ will be termed equivalent to $\mathcal{I}_2$ ($\mathcal{I}_1 \sim \mathcal{I}_2$) if and only if $A_{\mathcal{I}_1} = A_{\mathcal{I}_2}$.

It is easy to see that $\sim$ as defined above is an equivalence relation.

Using Lemma 1, we can easily deduce that $\mathcal{I}_1 \sim \mathcal{I}_2$ if and only if $\mathcal{I}_1$ can be obtained from $\mathcal{I}_2$ using a finite number of allowable (in the sense of Lemma 1) transpositions. It is clear that every equivalent class of index tuples defines uniquely one product of elementary matrices and vice versa.

Definition 3 Let $k, l \in \mathbb{Z}$ with $k \leq l$. Then we define

$$(k:l) := \begin{cases} (k, k+1, ..., l), k \leq l \\ \varnothing, k > l \end{cases} \tag{8}$$

Remark 1 According to the above definition, if $k_1 > l$ and $k_2 > l$, then both index tuples $(k_1 : l)$ and $(k_2 : l)$ correspond to the empty index tuple. To avoid this notational ambiguity in what follows, we shall adopt the notation $(\infty : l)$ for any tuple of the form $(k : l)$ having $k > l$ where applicable.

Definition 4 Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two tuples. By $(\mathcal{I}_1, \mathcal{I}_2)$ we denote the juxtaposition of $\mathcal{I}_1$ and $\mathcal{I}_2$.

In view of the previous definition it is clear that

$$A_{(\mathcal{I}_1,\mathcal{I}_2)} = A_{\mathcal{I}_1} A_{\mathcal{I}_2}.$$

In general juxtaposition of index tuples is not commutative, i.e. $(\mathcal{I}_1, \mathcal{I}_2) \nsim (\mathcal{I}_2, \mathcal{I}_1)$. Furthermore, by noticing that $A_{(\mathcal{I}_1,\varnothing)} = A_{\mathcal{I}_1}$ we shall adopt the convention

$$A_{\varnothing} = I_{np}.$$

Definition 5 Given an index tuple $\mathcal{I} = (i_1, i_2, \ldots, i_m)$. We define the reverse tuple $(i_m, i_{m-1}, \ldots, i_1)$, which will be denoted as $\bar{\mathcal{I}}$.

Clearly, the reverse operator satisfies the following property

$$\overline{(\mathcal{I}_1, \mathcal{I}_2)} \sim (\bar{\mathcal{I}}_2, \bar{\mathcal{I}}_1).$$

We now introduce a very important notion that will play a central role for the construction of linearizations proposed in the next section.

Definition 6 A product of two elementary matrices A_i, A_j with $i, j \in \{0, 1, 2, \ldots, n-1\}$ will be called *operation free* iff the block elements of the product are either $0, I_p$ or $-T_i$ (for generic matrices T_i).

Obviously a product consisting of just one term A_i is in the above sense operation free.

Lemma 2 *The product $A_i A_i$ is not operation free for $i = 0, \ldots, n-1$.*

Proof It is easy to see that for $i = 0$

$$A_0 A_0 = \begin{bmatrix} T_0^2 & \\ & I_{p(n-1)} \end{bmatrix}$$

and for $0 < i \leq n-1$

$$A_i A_i = \begin{bmatrix} I_{p(i-1)} & & & \\ & I_p & -T_i & \\ & -T_i & T_i^2 + I_p & \\ & & & I_{p(n-i-1)} \end{bmatrix}.$$

The definition of operation free products can be inductively extended to products of more than two elementary matrices. However, it can be easily confirmed that the product of two operation free products is not necessarily operation free itself. This is the case in the second product in the following lemma.

Lemma 3 *The product $A_iA_{i+1}A_i$ is operation free, while $A_{i+1}A_iA_{i+1}$ is not for $i = 0, ..., n-2$.*

Proof For $i = 0$

$$A_0A_1A_0 = \begin{bmatrix} 0 & -T_0 & \\ -T_0 & -T_1 & \\ & & I_{p(n-2)} \end{bmatrix}$$

is operation free. The second product for $i = 0$ is

$$A_1A_0A_1 = \begin{bmatrix} I_p & -T_1 & \\ -T_1 & T_1^2 - T_0 & \\ & & I_{p(n-2)} \end{bmatrix}$$

which is not operation free for generic matrices T_i. For $0 < i \leq n-2$

$$A_iA_{i+1}A_i = \begin{bmatrix} I_{p(i-1)} & & & & \\ & 0 & 0 & I_p & \\ & 0 & I_p & -T_i & \\ & I_p & -T_i & -T_{i+1} & \\ & & & & I_{p(n-i-2)} \end{bmatrix}$$

is obviously operation free, while

$$A_{i+1}A_iA_{i+1} = \begin{bmatrix} I_{p(i-1)} & & & & \\ & 0 & 0 & 1 & \\ & 0 & 1 & -T_{i+1} & \\ & 1 & -T_{i+1} & T_{i+1}^2 - T_i & \\ & & & & I_{p(n-i-2)} \end{bmatrix}$$

is not.

An important property of non operation free products is that they can not be extended to operation free ones as shown below.

Lemma 4 *Let $\mathcal{M}$ be an index tuple such that $A_{\mathcal{M}}$ is not operation free. Then for any two other index tuples $\mathcal{L}$ and $\mathcal{R}$, $A_{\mathcal{L}}A_{\mathcal{M}}A_{\mathcal{R}}$ is not operation free.*

Proof Assume that $A_{\mathcal{M}}$ is not operation free and let $M_{ij} \in \mathbb{C}^{p\times p}$, $i, j = 1, 2, \ldots, n$ be its block elements. Since $A_{\mathcal{M}}$ is a product of elementary matrices $A_k, k = 0, 1, \ldots, n-1$ defined by (7), (5), it is convenient to set $S_k = -T_k$, only to make it easier to see that if operations occur in an element M_{ij}, these can only be additions and multiplications between blocks 0, I_p and (generic) S_k's.

We show first that $A_{\mathcal{M}}A_{\mathcal{R}}$ is not operation free. Let $\mathcal{R} = (r_1, r_2, \ldots, r_v)$ and consider the product $A_{\mathcal{M}}A_{r_1}$. In view of the definition of the elementary matrices (7) and (5) we distinguish two cases

- If $r_1 = 0$, the block columns 2 to n of $A_{\mathcal{M}}$ remain intact after the post-multiplication by A_0, while the blocks of the first column become $M_{i,1}S_0$. Obviously, if any of the block elements in columns 2 to n is not operation free in $A_{\mathcal{M}}$, so will be in $A_{\mathcal{M}}A_0$. On the other hand, if operations occur in some element $M_{i,1}$, in the first column of $A_{\mathcal{M}}$, these can only be additions and multiplications between blocks 0, I_p and S_k. Hence, $M_{i,1}S_0$ will still involve operations (for generic S_0).

– If $r_1 = 1, 2, \ldots, n-1$, by carrying out the multiplication $A_{\mathcal{M}} A_{r_1}$ we notice that the effect of the post multiplication by A_{r_1}, is only limited to block columns r_1 and $r_1 + 1$. Particularly the block elements R_{ij} of $A_{\mathcal{M}} A_{r_1}$, have the form

$$R_{ij} = \begin{cases} M_{ij}, & j \neq r_1 \text{ and } j \neq r_1 + 1 \\ M_{i,r_1+1}, & j = r_1 \\ M_{i,r_1} + M_{i,r_1+1} S_{r_1}, & j = r_1 \end{cases}$$

In view of the above, if operations occur in any column, other than r_1 in $A_{\mathcal{M}}$, so will be in $A_{\mathcal{M}} A_{r_1}$ and there is nothing to prove further. Assume now that operations occur only in block column r_1 of $A_{\mathcal{M}}$ and consider an element M_{i,r_1} in this column which containing operations. Since, operations occur only in block column r_1 of $A_{\mathcal{M}}$, M_{i,r_1+1} is operation free and it may take one of the values 0, I_p or S_j for some j. Depending on the value of M_{i,r_1+1} we have

– If $M_{i,r_1+1} = 0$ then $R_{i,r_1+1} = M_{i,r_1}$ so operations still occur in R_{i,r_1+1}.
– If $M_{i,r_1+1} = I_p$ then $R_{i,r_1+1} = M_{i,r_1} + S_{r_1}$. Since, M_{i,r_1} involves only additions and multiplications of 0, I_p and S_k's, the existence of a term of the form $-S_k$ in M_{i,r_1}, is not possible. Thus, the generic block S_{r_1} cannot cause a cancellation in R_{i,r_1+1}, since this would require the presence of a term of the form $-S_{r_1}$ in M_{i,r_1}. Thus, M_{i,r_1+1} is not operation free.
– If $M_{i,r_1+1} = S_j$ then $R_{i,r_1+1} = M_{i,r_1} + S_j S_{r_1}$. Similarly to the previous case, $S_j S_{r_1}$ can only cancel a term of the form $-S_j S_{r_1}$ in M_{i,r_1}, which is can not exist as explained above. Again, M_{i,r_1+1} is not operation free.

So far we have shown that $A_{\mathcal{M}} A_{r_1}$ can not be operation free for any r_1. Applying successively the same argument for the multiplication $A_{\mathcal{M}} A_{r_1}$ by A_{r_2} and respectively for the product $A_{\mathcal{M}} A_{r_1} A_{r_2}$ by A_{r_3} etc., we conclude that $A_{\mathcal{M}} A_{r_1} A_{r_2} \ldots A_{r_v} = A_{\mathcal{M}} A_{\mathcal{R}}$ is not operation free.

Using a similar procedure we can show the corresponding result for the pre-multiplication of $A_{\mathcal{M}} A_{\mathcal{R}}$ by $A_{\mathcal{L}}$, which finally proves that $A_{\mathcal{L}} A_{\mathcal{M}} A_{\mathcal{R}}$ is not operation free.

An immediate consequence of the above Lemma is that if $A_{\mathcal{I}}$ is an operation free product of elementary matrices and $\mathcal{M}$ is any index tuple such that $\mathcal{I} = (\mathcal{L}, \mathcal{M}, \mathcal{R})$ for some tuples $\mathcal{L}, \mathcal{R}$, then $A_{\mathcal{M}}$ is operation free.

We now introduce the notion of block transposition of a block matrix. If $A = [A_{ij}]_{n \times m}$ is block matrix consisting of block elements $A_{ij} \in \mathbb{C}^{p \times p}$, then its *block transpose* is defined by

$$A^{\mathcal{B}} = [A_{ji}]_{m \times n}$$

Obviously, block transposition coincides with ordinary transposition for matrices consisting of scalar elements A_{ij}. It is important to notice that the well known property $(AB)^T = B^T A^T$, does not hold in general in the case of block transposition. However, this property holds for operation free products of elementary matrices as shown in the following.

Lemma 5 *Let $A_{\mathcal{I}}, A_{\mathcal{J}}$ be two products of elementary matrices whose product $A_{\mathcal{I}} A_{\mathcal{J}}$ is operation free. Then*

$$(A_{\mathcal{I}} A_{\mathcal{J}})^{\mathcal{B}} = A_{\mathcal{J}}^{\mathcal{B}} A_{\mathcal{I}}^{\mathcal{B}} \tag{9}$$

Proof As a direct consequence of Lemma 4, if $A_{\mathcal{I}}A_{\mathcal{J}}$ is operation free then $A_{\mathcal{I}}$, $A_{\mathcal{J}}$ will be operation free as well. Let $A_{\mathcal{I}} = [L_{ij}]_{n\times n}$, $A_{\mathcal{J}} = [R_{ij}]_{n\times n}$, $(A_{\mathcal{I}}A_{\mathcal{J}})^{\mathcal{B}} = [P_{ij}]_{n\times n}$ and $A_{\mathcal{J}}^{\mathcal{B}}A_{\mathcal{I}}^{\mathcal{B}} = [Q_{ij}]_{n\times n}$ where L_{ij}, R_{ij}, P_{ij} and Q_{ij} are the block elements of $A_{\mathcal{I}}$, $A_{\mathcal{J}}$, $(A_{\mathcal{I}}A_{\mathcal{J}})^{\mathcal{B}}$ and $A_{\mathcal{J}}^{\mathcal{B}}A_{\mathcal{I}}^{\mathcal{B}}$ respectively. It can be easily seen that

$$P_{ij} = \sum_{k=1}^{n} L_{jk}R_{ki},\; Q_{ij} = \sum_{k=1}^{n} R_{ki}L_{jk}$$

Obviously, if the block elements L_{jk}, R_{ki} commute for every $i, j, k,$, then $P_{ij} = Q_{ij}$ and (9) has been shown. Since both $A_{\mathcal{I}}, A_{\mathcal{J}}$ are operation free, their block elements L_{jk}, R_{ki} can be either $0, I_p$ or $-T_v$. Clearly, 0 and I_p commute with every matrix, so L_{jk}, R_{ki} do commute if at least one of them is either 0, I_p. If on the other hand, both L_{jk}, R_{ki} are of the form $-T_v$ and $-T_\mu$, which do not commute in general. However, this last case can not occur since $A_{\mathcal{I}}A_{\mathcal{J}}$ has been assumed to be operation free and the presence of a term of the T_vT_μ in some P_{ij}, can not be canceled (for details see proof of Lemma 4). Hence, if $A_{\mathcal{I}}A_{\mathcal{J}}$ is operation free L_{jk}, R_{ki} commute for every i, j, k and (9) has been proved.

Using associativity, the result of Lemma 5 can be extended to products of more than two factors. Making use of this extended version of Lemma 5 and taking into account the fact that elementary matrices are block symmetric (that is $A_k^{\mathcal{B}} = A_k$), we can easily confirm that if $A_{\mathcal{I}}$ is an operation free product then

$$A_{\bar{\mathcal{I}}} = A_{\mathcal{I}}^{\mathcal{B}} \tag{10}$$

which also an operation free product. Note that an operation free product $A_{\mathcal{I}}$ is block symmetric if and only

$$A_{\mathcal{I}} = A_{\mathcal{I}}^{\mathcal{B}} = A_{\bar{\mathcal{I}}}$$

i.e. if and only if

$$\mathcal{I} \sim \bar{\mathcal{I}}. \tag{11}$$

Lemma 6 *The product $A_{(k:l)}$ is of the form*

$$A_{(k:l)} = \begin{cases} \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & 0_{1\times(l-k+1)} & I & \\ \cline{2-3} & & -T_k & \\ & I_{l-k+1} & \vdots & \\ & & -T_l & \\ \hline & & & I_{n-l-1} \end{array}\right], k>0 \\ \left[\begin{array}{c|c|c} 0_{1\times l} & -T_0 & \\ \hline & -T_1 & \\ I_l & \vdots & \\ & -T_l & \\ \hline & & I_{n-l-1} \end{array}\right], k=0 \end{cases}, k \le l \le n-1 \tag{12}$$

where the dimensions appearing in the zero and identity matrices are block dimensions.

Proof We first deal with the case $k > 0$.We shall use induction on l. For $l = k$, $A_{(k:k)} = A_k$ which is in accordance with (12). Assuming that (12) holds for l we shall prove that $A_{(k:l+1)}$ is also given by (12). Indeed

$$
\begin{aligned}
A_{(k:l+1)} &= A_{(k:l)} A_{l+1} \\
&= \left[\begin{array}{c|cc|c}
I_{k-1} & & & \\ \hline
 & 0_{1\times(l-k+1)} & I & \\ \cline{2-3}
 & & -T_k & \\
 & I_{(l-k+1)} & \vdots & \\
 & & -T_l & \\ \hline
 & & & I_{n-l-1}
\end{array}\right]
\left[\begin{array}{c|cc|c}
I_l & & & \\ \hline
 & 0 & I & \\
 & I & -T_{l+1} & \\ \hline
 & & & I_{n-l-2}
\end{array}\right] \\
&= \left[\begin{array}{c|cc|c}
I_{k-1} & & & \\ \hline
 & 0_{1\times(l-k+2)} & I & \\ \cline{2-3}
 & & -T_k & \\
 & I_{(l-k+2)} & \vdots & \\
 & & -T_{l+1} & \\ \hline
 & & & I_{n-l-2}
\end{array}\right]
\end{aligned}
$$

which is in conforms with (12). The case $k = 0$ follows easily using the fact that $A_{(0:l)} = A_0 A_{(1:l)}$.

Notice that the matrices T_i appear in the $(l + 1)$ block column while in each of the remaining block columns appear exactly one identity matrix and zeros. Note also that every product $A_{(k:l)}$ is operation free. The following Theorem provides a canonical form for the expression of products of elementary matrices A_i.

Theorem 1 *Every product of the form*

$$\prod_{i=n-1}^{0} A_{(c_i:i)}, \textit{ for } c_i \in (0 : i) \cup \{\infty\} \tag{13}$$

is an operation free product. Form (13) of a product of elementary matrices will be called column standard form.

Proof In order to prove the theorem we will use induction on n in (13).
The theorem obviously holds for $n = 1$.
Assume that for some $n > 1$, the product $\prod_{i=n-1}^{0} A_{(c_i:i)}$ is operation free , for $c_i \in (0 : i) \cup \{\infty\}$.
For dimension $n + 1$ we define $\bar{A}_i, i = 0, ..., n$ the corresponding elementary matrices. We will prove that $\prod_{i=n}^{0} \bar{A}_{(c_i:i)}$, for $c_i \in (0 : i) \cup \{\infty\}$ is operation free.

$$
\begin{aligned}
\prod_{i=n}^{0} \bar{A}_{(c_i:i)} = \bar{A}_{(c_n:n)} \prod_{i=n-1}^{0} \bar{A}_{(c_i:i)} = \bar{A}_{(c_n:n)} \prod_{i=n-1}^{0} \bar{A}_{(c_i:i)} = \\
= \bar{A}_{(c_n:n)} \prod_{i=n-1}^{0} diag\{A_{(c_i:i)}, I_p\}.
\end{aligned} \tag{14}
$$

From Lemma 6 follows that $\bar{A}_{(c_n:n)}$ has T_i 's only in the last column. Due to the block diagonal structure of $\prod_{i=n-1}^{0} diag\{A_{(c_i:i)}, I_p\}$, we can conclude that (14) is operation free.

Using the notational convention of remark 1, in the choice of base indices c_i, the products of the form (13) are completely characterized by the ordered set of indices $\mathcal{C} = (c_{n-1}, c_{n-2}, \ldots, c_0)$.

Example 1 Let $T(s)$ a $p \times p$ polynomial matrix of degree $n = 6$. Consider the product

$$\tilde{A} = (A_3A_4A_5)(A_3A_4)(A_3)(A_0A_1A_2)(A_0). \tag{15}$$

Rewriting $\tilde{A}$ as

$$\tilde{A} = A_{(3:5)}A_{(3:4)}A_{(3:3)}A_{(0:2)}A_{(\infty:1)}A_{(0:0)}, \tag{16}$$

we notice that the above product is in the standard column form described in (13) and thus it is operation free. The matrix $\tilde{A}$ is characterized by the base index tuple $\mathcal{C} = (3, 3, 3, 0, \infty, 0)$.

Note that the form of the product (16) indicates the block column of $\tilde{A}$ on which an element T_i resides (see proof of Theorem 1). The i-th column will contain exactly those T_i's with $i \in (c_{i-1} : i-1)$ corresponding to the indices in the product (16). This fact is summarized in the following table.

# column	1	2	3	4	5	6
T_i	T_0		T_0, T_1, T_2	T_3	T_3, T_4	T_3, T_4, T_5

(17)

After carrying out the products, we have that

$$\tilde{A} = \begin{bmatrix} 0 & 0 & -T_0 & 0 & 0 & 0 \\ -T_0 & 0 & -T_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_p \\ 0 & 0 & 0 & 0 & I_p & -T_3 \\ 0 & 0 & 0 & I_p & -T_3 & -T_4 \\ 0 & I_p & -T_2 & -T_3 & -T_4 & -T_5 \end{bmatrix}.$$

Example 2 Let $T(s)$ a $p \times p$ polynomial matrix of degree $n = 4$. The matrix that corresponds to $\mathcal{C} = (0, 0, 0, 0)$ is the block symmetric matrix

$$\tilde{A} = \prod_{i=n-1}^{0} A_{(0:i)} = \begin{bmatrix} 0 & 0 & 0 & -T_0 \\ 0 & 0 & -T_0 & -T_1 \\ 0 & -T_0 & -T_1 & -T_2 \\ -T_0 & -T_1 & -T_2 & -T_3 \end{bmatrix}.$$

Example 3 The matrix with column standard form corresponding to $\mathcal{C} = (0, \infty, \infty, \infty, ...\infty)$ is the companion matrix $\hat{P}_0$ in (4) while the one corresponding to $\mathcal{C} = (n-1, ..., 2, 1, 0)$ is the companion matrix P_0 in (3).

Lemma 7 *There exist* $(n + 1)!$ *distinct products of the form (13).*

Proof Each product of the form (13) is completely characterized by the ordered set of indices $\mathcal{C} = (c_{n-1}, c_{n-2}, \ldots, c_0)$. The index c_{n-1} ranges from 0 to $n-1$, i.e. it may take n distinct values, but we also allow it to take the value ∞ (which corresponds to the empty set). Totally the index c_{n-1} may take $n+1$ distinct values. Similarly, c_{n-2} may take n distinct values, c_{n-3}, $n-1$ distinct values and so on. Thus, there exist $(n+1)!$ distinct choices of $\mathcal{C}$.

We introduce now the successor infix property of index tuples, which plays a crucial role in the characterization of operation free products as shown in the sequel.

Definition 7 (Successor Infix Property (SIP)) Let $\mathcal{I} = (i_1, i_2, \ldots, i_k)$ be an index tuple. $\mathcal{I}$ will be called successor infixed if and only if for every pair of indices $i_a, i_b \in \mathcal{I}$, with $1 \leq a < b \leq k$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$, such that $a < c < b$.

According to the above definition, if $\mathcal{I}$ satisfies the SIP, then for any partitioning of $\mathcal{I} = (\mathcal{L}, \mathcal{M}, \mathcal{R})$ the tuples $\mathcal{L}, \mathcal{M}$ and $\mathcal{R}$ will also satisfy the SIP. It is also clear from the above definition that SIP imposes certain limitations on the number of occurrences of each index in the tuple, depending on its value. For instance, since $n-1$ is the greatest amongst all indices, it should appear only once in the tuple. Consequently, there can be at most two instances of the index $n-2$, since the SIP requires them to be separated by an index of value $n-1$ and the maximal index appears only once. Proceeding inductively it is easy to see that an SIP index tuple corresponding to a polynomial matrix of degree n, can have at most $n(n+1)/2$ indices. Before proceeding to the main result of this section, we state and prove an auxiliary lemma.

Lemma 8 *Let $\mathcal{I}$ be an index tuple satisfying the SIP and let m be the maximal index in $\mathcal{I}$. Then there exists an index $c_m \leq m$ such that*

$$\mathcal{I} \sim ((c_m : m), \mathcal{I}') \tag{18}$$

with $m \notin \mathcal{I}'$.

Proof Since $\mathcal{I}$ satisfies the SIP and m is its maximal index, there can be only one occurrence of m in $\mathcal{I}$. Otherwise, between any two instances of m there should be an index with value $m+1$, which contradicts the maximality of m. Since there is only one instance of m we can partition $\mathcal{I}$ as follows

$$\mathcal{I} = (\mathcal{L}_0, m, \mathcal{R}_0)$$

where $m \notin \mathcal{L}_0$ and $m \notin \mathcal{R}_0$. Taking into account the SIP we notice that there can be at most one occurrence of the index $m-1$ in each of the tuples $\mathcal{L}_0, \mathcal{R}_0$. Otherwise, if $m-1$ appears twice (or more) in either of the tuples $\mathcal{L}_0, \mathcal{R}_0$, the SIP is violated since there is no index m in $\mathcal{L}_0$ and $\mathcal{R}_0$. We distinguish two cases:

– If $m-1 \notin \mathcal{L}_0$, then according to Lemma 1, m commutes with every index in $\mathcal{L}_0$ and we can write
$$\mathcal{I} \sim (m, \mathcal{L}_0, \mathcal{R}_0)$$
In this case (18) has been obtained, by setting $c_m = m$, $\mathcal{I}' = (\mathcal{L}_0, \mathcal{R}_0)$ and noticing that $m \notin \mathcal{L}_0, \mathcal{R}_0$.

– If $m-1 \in \mathcal{L}_0$, using the fact that $m-1$ appears at most once in $\mathcal{L}_0$, we can partition $\mathcal{L}_0 = (\mathcal{L}_1, m-1, \mathcal{L}'_0)$ so that $m-1 \notin \mathcal{L}_1$ and $m-1 \notin \mathcal{L}'_0$. Now m commutes with the elements of $\mathcal{L}'_0$ and we can write $\mathcal{I} \sim (\mathcal{L}_1, m-1, m, \mathcal{L}'_0, \mathcal{R}_0)$. Thus setting $\mathcal{R}_1 = (\mathcal{L}'_0, \mathcal{R}_0)$ we can write

$$\mathcal{I} \sim (\mathcal{L}_1, (m-1:m), \mathcal{R}_1)$$

where $m, m-1 \notin \mathcal{L}_1$, $m \notin \mathcal{R}_1$ and because of the SIP, $m-2$ appears at most once in $\mathcal{L}_1$.

Proceeding accordingly, depending on the existence of $m-2$ in $\mathcal{L}_1$ and taking into account the SIP, we can write either

$$\mathcal{I} \sim ((m-1:m), \mathcal{L}_1, \mathcal{R}_1)$$

which gives the form (18) for $c_m = m-1$, $\mathcal{I}' = (\mathcal{L}_1, \mathcal{R}_1)$ (notice that $m \notin \mathcal{L}_1, \mathcal{R}_1$) or

$$\mathcal{I} \sim (\mathcal{L}_2, (m-2:m), \mathcal{R}_2).$$

In view of the above reduction, it is clear that $\mathcal{L}_{i+1}$ has strictly less elements than $\mathcal{L}_i$.Thus the same procedure can be applied for a finite number of times, until $\mathcal{L}_i$ vanishes, so that (18) is obtained.

Theorem 2 *Let $\mathcal{I}$ be an index tuple. The following statements are equivalent*

1. *$A_\mathcal{I}$ is operation free*
2. *$\mathcal{I}$ satisfies the SIP*
3. *$A_\mathcal{I}$ can be written in the column standard form (13) as $\prod_{i=n-1}^{0} A_{(c_i:i)}$, for $c_i \in (0:i) \cup \{\infty\}$.*

Proof $(1 \Rightarrow 2)$ Assume $A_\mathcal{I}$ is operation free and that $\mathcal{I}$ does not satisfy the SIP. We will show that this is a contradiction. Choose the minimum index $i_0 \in \mathcal{I}$, not satisfying the SIP. Furthermore, we can always choose two $i_0 \in \mathcal{I}$, such that $\mathcal{I}$ can be partitioned as follows

$$\mathcal{I} = (\mathcal{L}, i_0, \mathcal{M}, i_0, \mathcal{R}) \tag{19}$$

with $\mathcal{M}$ some index set with $i_0, i_0+1 \notin \mathcal{M}$. Then we distinguish two cases:

– $i_0 - 1 \notin \mathcal{M}$. Then by Lemma 1, i_0 commutes with every element of $\mathcal{M}$ and thus can write $\mathcal{I} \sim (\mathcal{L}, \mathcal{M}, i_0, i_0, \mathcal{R})$ .From lemma 2 it is clear that the product $A_{(i_0,i_0)}$ is not operation free. Hence, according to lemma 4, $A_\mathcal{I}$ is not operation free.
– $i_0 - 1 \in \mathcal{M}$. There can be only one occurrence of $i_0 - 1$ in $\mathcal{M}$, since otherwise $i_0 - 1$ would be the minimum integer not satisfying the SIP. Thus, $\mathcal{M} = (\mathcal{M}_L, i_0 - 1, \mathcal{M}_R)$ where $i_0 - 1 \notin \mathcal{M}_R$ and $i_0 - 1 \notin \mathcal{M}_L$. Using Lemma 1 we have $\mathcal{I} \sim (\mathcal{L}, \mathcal{M}_L, i_0, (i_0-1), i_0, \mathcal{M}_R, \mathcal{R})$. Then $A_\mathcal{I} = A_{(\mathcal{L},\mathcal{M}_L)} A_{(i_0,i_0-1,i_0)} A_{(\mathcal{M}_R,\mathcal{R})}$ is not operation free in view of Lemma 3 in conjunction with lemma 4.

In both cases we have arrived at a contradiction. Hence, $\mathcal{I}$ satisfies the SIP.

$(2 \Rightarrow 3)$ Assume that $\mathcal{I}$ satisfies the SIP and let m_0 be the maximal index in $\mathcal{I}$. Using Lemma 8 we obtain that

$$\mathcal{I} \sim ((c_{m_0} : m_0), \mathcal{I}_1)$$

where $m_0 \notin \mathcal{I}_1$. Assuming that m_1 is the maximal index in $\mathcal{I}_1$ and reapplying Lemma 8 we obtain

$$\mathcal{I} \sim ((c_{m_0} : m_0), (c_{m_1} : m_1), \mathcal{I}_2)$$

where $m_0, m_1 \notin \mathcal{I}_2$.Obviously $m_0 > m_1$. Proceeding accordingly we obtain

$$\mathcal{I} \sim ((c_{m_0} : m_0), (c_{m_1} : m_1), \ldots, (c_{m_p} : m_p))$$

where $m_0 > m_1 > \ldots > m_p$. The matrix product corresponding to $\mathcal{I}$ is

$$A_{\mathcal{I}} = A_{(c_{m_0}:m_0)} A_{(c_{m_1}:m_1)} \cdots A_{(c_{m_p}:m_p)}$$

which is in column standard form if we take into account Remark 1 and introduce terms of the form $A_{(\infty:i)}$ for indices $i \notin \{m_0, m_1, \ldots, m_p\}$.

(3 ⇒ 1) This follows directly from theorem 1.

Theorem 2 can actually be extended to include a fourth equivalent statement, dual to the third one. Given an operation free product $A_{\mathcal{I}}$, we consider its block transpose which by Lemma 5 is given by

$$A_{\mathcal{I}}^{\mathcal{B}} = A_{\bar{\mathcal{I}}}$$

Obviously, $A_{\bar{\mathcal{I}}}$ is an operation free product itself, so according to theorem 2 it can be written in its column standard form

$$A_{\bar{\mathcal{I}}} = \prod_{i=n-1}^{0} A_{(r_j:j)}, \text{ for } r_j \in (0:j) \cup \{\infty\}$$

where we have used indices r_i, instead of c_i. Now, reapply block transposition to $A_{\mathcal{I}}^{\mathcal{B}}$ to see that

$$\left(A_{\mathcal{I}}^{\mathcal{B}}\right)^{\mathcal{B}} = A_{\bar{\mathcal{I}}}^{\mathcal{B}} = \left(\prod_{i=n-1}^{0} A_{(r_j:j)}\right)^{\mathcal{B}} = \prod_{j=0}^{n-1} A_{(r_i:i)}^{\mathcal{B}} = \prod_{j=0}^{n-1} A_{\overline{(r_j:j)}}$$

But $\left(A_{\mathcal{I}}^{\mathcal{B}}\right)^{\mathcal{B}} = A_{\mathcal{I}}$, so

$$A_{\mathcal{I}} = \prod_{j=0}^{n-1} A_{\overline{(r_j:j)}}, \text{ for } r_j \in (0:j) \cup \{\infty\} \tag{20}$$

The form (20) will be termed *row standard form* of the operation free product $A_{\mathcal{I}}$. So every operation free product $A_{\mathcal{I}}$ can be written in row standard form. Reversely, the product on the r.h.s. of (20) is clearly the block transpose of a column standard form, thus an operation free product. Thus, $A_{\mathcal{I}}$ is operation free if and only if it can be written in row standard form.

Example 4 The matrix $\tilde{A} = (A_3A_4A_5)(A_3A_4)(A_3)(A_0A_1A_2)(A_0)$ in example 1, using the commutative properties of A_i can be written as

$$\begin{aligned}\tilde{A} &= A_3A_4\underrightarrow{A_5}A_3A_4A_3A_0A_1A_2A_0 = A_3\underrightarrow{A_4}A_3A_0A_1A_0(A_5A_4A_3A_2) = \\ &= \underrightarrow{A_3}A_0A_1A_0(A_4A_3)(A_5A_4A_3A_2) = (A_0)(A_1A_0)(A_3)(A_4A_3)(A_5A_4A_3A_2)\end{aligned} \tag{21}$$

which is in the row standard form (20). The arrow below the matrices denote that in the next step the corresponding matrix will be forwarded as much to the right as possible using the commutative properties of A_i. From the form of (21) we have the following table concerning the rows of $\tilde{A}$.

# row	T_i
1	T_0
2	T_0, T_1
3	
4	T_3
5	T_3, T_4
6	T_2, T_3, T_4, T_5

(22)

Using the tables (17) and (22), the fact that the rows of $\tilde{A}$ contain T_i in an ascending order from left to right while the columns contain T_i in an ascending order from top to bottom, one can deduce the positions of T_i's without the need of carrying out the multiplications.

Having explored a wide range of properties regarding operation free products of A_k with $k \in \{0, \dots, n-1\}$, we can note that similar results can be obtained using products of A_k^{-1} which have a particularly simple form, that is

$$A_{-k} := A_k^{-1} = \begin{bmatrix} I_{p(k-1)} & 0 & \cdots \\ 0 & C_k^{-1} & \ddots \\ \vdots & \ddots & I_{p(n-k-1)} \end{bmatrix}, \; k = 1, \dots, n-1 \tag{23}$$

with

$$C_{-k} := C_k^{-1} = \begin{bmatrix} T_k & I_p \\ I_p & 0 \end{bmatrix}.$$

while we define A_{-n} to be

$$A_{-n} = diag\{I_{p(n-1)}, T_n\}.$$

The definitions and properties regarding the index tuples can be naturally extended to cover indices from the set $\{-n, ..., -1\}$.

It can be shown that products $A_{(-k:-l)}$ where $1 \le l \le k \le n$ have the form

$$A_{(-k:-l)} = \begin{cases} \left[\begin{array}{c|c|c|c} I_{l-1} & & & \\ \hline & T_l & & \\ & \vdots & I_{(k-l+1)} & \\ & T_k & & \\ \cline{2-3} & I & 0_{1\times(k-l+1)} & \\ \hline & & & I_{n-k-1} \end{array}\right], & l \le k < n \\ \left[\begin{array}{c|c|c} I_{l-1} & & \\ \hline & T_l & \\ & \vdots & I_{(n-l)} \\ & T_{n-1} & \\ \hline & T_n & 0_{1\times(n-l)} \end{array}\right], & k = n \end{cases}$$

where the dimensions appearing in the zero and identity matrices are block dimensions.

In the following we generalize the definitions of operation free products and SIP to deal with index tuples with elements in $\{-n, ..., -1\}$.

Definition 8 A product of two elementary matrices A_i, A_j with $i, j \in \{-n, ..., -1\}$ will be called *operation free* iff the block elements of the product are either $0, I_p$ or T_i (for generic matrices T_i).

Definition 9 Let $\mathcal{I} = (i_1, i_2, \ldots, i_k)$ be an index tuple with elements from $\{-n, ..., -1\}$. $\mathcal{I}$ will be called successor infixed if and only if for every pair of indices $i_a, i_b \in \mathcal{I}$, with $1 \leq a < b \leq k$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$, such that $a < c < b$.

Using similar arguments as in the beginning of the section, the following theorem holds.

Theorem 3 *Let $\mathcal{I} = (i_1, i_2, \ldots, i_m)$ be an index tuple from the set $\{-n, ..., -1\}$. The following statements are equivalent*

1. *$A_{\mathcal{I}}$ is operation free*
2. *$\mathcal{I}$ satisfies the SIP*
3. *$A_{\mathcal{I}}$ can be written in the column standard form $\prod_{i=-1}^{-n} A_{(c_i:i)}$, for $c_i \in (-n : i) \cup \infty$.*
4. *$A_{\mathcal{I}}$ can be written in the row standard form $\prod_{j=-n}^{-1} A_{\overline{(r_j:j)}}$, for $r_j \in (-n : j) \cup \infty$.*

Example 5 Let $T(s)$ a $p \times p$ polynomial matrix of degree $n = 6$. Then the product

$$\tilde{A} = A_{-4}A_{-3}A_{-5}A_{-2}A_{-1}A_{-4}A_{-3}A_{-2}A_{-5}A_{-4}A_{-5}A_{-6}$$

is an operation free product. $\tilde{A}$ can be rewritten in the column standard form as

$$\tilde{A} = A_{(-4:-1)}A_{(-5:-2)}IA_{(-5:-4)}A_{(-5:-5)}A_{(-6:-6)}.$$

or in the row standard form as

$$\begin{aligned}\tilde{A} &= A_{\overline{(-5:-4)}}A_{\overline{(-5:-3)}}A_{\overline{(-6:-2)}}A_{\overline{(-2:-1)}} = \\ &= (A_{-4}A_{-5})(A_{-3}A_{-4}A_{-5})(A_{-2}A_{-3}A_{-4}A_{-5}A_{-6})(A_{-1}A_{-2}).\end{aligned}$$

After carrying out the operations, we can see that

$$\tilde{A} = \begin{bmatrix} T_1 & T_2 & I & 0 & 0 & 0 \\ T_2 & T_3 & 0 & T_4 & T_5 & T_6 \\ T_3 & T_4 & 0 & T_5 & I & 0 \\ T_4 & T_5 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is indeed operation free.

Example 6 Let $T(s)$ a $p \times p$ polynomial matrix of degree $n = 4$. The matrix $\tilde{A} = \prod_{i=-1}^{-n} A_{(-n:i)}$ is the symmetric matrix

$$\tilde{A} = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \\ T_2 & T_3 & T_4 & 0 \\ T_3 & T_4 & 0 & 0 \\ T_4 & 0 & 0 & 0 \end{bmatrix}.$$

The next Lemma follows easily as an extension of Lemma 1.

Lemma 9 *Let A_i and A_j two elementary matrices with $i, j \in \{-n, ..., -1, 0, \ldots, n-1\}$. Then if $||i| - |j|| \neq 1$ the two elementary matrices commute i.e. $A_i A_j = A_j A_i$.*

3 Matrix pencil linearizations

First order representations of a polynomial matrix are of particular interest in many research fields such as eigenvalue problems, systems and control theory etc. Out of the infinite number of possible matrix pencil linearizations, in practice only a small subset of them are typically used, the companion form linearizations. Companion forms of polynomial matrices $T(s)$ (or even scalar polynomials) are linearizations where both the first order coefficient and the constant term consisting of block matrices $0, \pm I_p$ or $\pm T_i$. Due to this property, companion forms are of particular interest in many research fields both as a theoretical or computational tool since they are in general easier to manipulate, provide better insight on the underlying problem and the lack operations between the coefficients T_i, guarantees that the numerical data of the original problem are not perturbed.

In the previous section we have defined a family of constant matrices derived from the coefficients of a regular polynomial matrix $T(s)$ using products of elementary matrices and provided criteria for those products to contain only block matrices of the form $0, I_p$ or $\pm T_i$. In this section we will use those matrices to construct a new family of companion like linearizations corresponding to a polynomial matrix $T(s)$. It is worth noting that using the criteria concerning operation free matrices presented in the previous section, one can easily prove that the coefficients of the linearizations in [1] and [2] are operation free. Before concentrating on the main topic of this section, we will need the following auxiliary results.

Lemma 10 *Let $\mathcal{I}$ be an index tuple having the SIP with elements from the set $\{0, 1, \ldots, k\}$ where $k \in \{1, 2, \ldots, n\}$. Then $A_{\mathcal{I}}$ is of the form $diag\{X, I_{n-k-1}\}$ where X is a $k+1 \times k+1$ matrix.*

Proof Since $\mathcal{I}$ has the SIP, $A_{\mathcal{I}}$ can be rewritten in the column standard form i.e. $A_{\mathcal{I}} = \prod_{i=k}^{0} A_{(c_i:i)}$. Since the structure of the matrix $A_{(c_i:i)}$ is known from (12), we can easily see that $A_{\mathcal{I}} = diag\{X, I_{n-k-1}\}$.

Using similar arguments and the column standard form of negative indices, the next Lemma can be proven.

Lemma 11 *Let $\mathcal{I}$ be an index tuple having the SIP with elements from the set $\{-n,-n+1,\ldots,-k\}$ where $k \in \{1,2,\ldots,n\}$. Then $A_{\mathcal{I}}$ is of the form $diag\{I_{k-1}, Y\}$ where Y is a $n-k+1 \times n-k+1$ matrix.*

Theorem 4 *Let $\mathcal{P}$ be a permutation of the index tuple $(0:k-1)$ where $k \in \{1,2,\ldots,n\}$. Let also $\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}}$ index tuples with elements from the set $\{0,1,\ldots,k-2\}$ such that $(\mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}})$ satisfies the SIP. Then $(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}})$ satisfies the SIP.*

Proof Assume that $(\mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}})$ satisfies the SIP while $(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}})$ does not. Then there must exist an index i such that between two of its occurrences in $(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}})$ no $i+1$ index exists. Those two occurrences of i cannot exist both in $\mathcal{L}_{\mathcal{P}}$ or $\mathcal{R}_{\mathcal{P}}$, because that would mean in view of Lemma 4 that $(\mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}})$ does not have the SIP. Then by choosing the rightmost problematic i in $\mathcal{L}_{\mathcal{P}}$ and the leftmost in $\mathcal{R}_{\mathcal{P}}$ we can write

$$\mathcal{L}_{\mathcal{P}} = (\mathcal{L}_{\mathcal{P}_1}, i, \mathcal{L}_{\mathcal{P}_2}),$$
$$\mathcal{R}_{\mathcal{P}} = (\mathcal{R}_{\mathcal{P}_1}, i, \mathcal{R}_{\mathcal{P}_2}),$$

where $i \notin \mathcal{L}_{\mathcal{P}_2}, \mathcal{R}_{\mathcal{P}_1}$. Then $(\mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}})$ can be rewritten as $(\mathcal{L}_{\mathcal{P}_1}, i, \mathcal{L}_{\mathcal{P}_2}, \mathcal{P}, \mathcal{R}_{\mathcal{P}_1}, i, \mathcal{R}_{\mathcal{P}_2})$. We then distinguish two cases.

- $i+1 \in \mathcal{P}$. Then by the definition of $\mathcal{P}$, $i \in \mathcal{P}$ also, which makes $(\mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}})$ not satisfying the SIP, which is a contradiction to the assumption.
- $i+1 \notin \mathcal{P}$. Then $(\mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}})$ does not satisfy the SIP, which is a contradiction to the assumption.

Thus the Theorem holds.

Theorem 5 *Let $\mathcal{N}$ be a permutation of the index tuple $(-n:-k)$ where $k \in \{1,2,\ldots,n\}$. Let also $\mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}}$ index tuples with elements from the set $\{-n,-n+1,\ldots,-k-1\}$ such that $(\mathcal{L}_{\mathcal{N}}, \mathcal{N}, \mathcal{R}_{\mathcal{N}})$ satisfies the SIP. Then $(\mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}})$ satisfies the SIP.*

Proof The proof is analogous to the one of Theorem 4.

Theorem 6 *Let $T(s)$ be a regular $p \times p$ polynomial matrix with degree p with T_0, T_n nonsingular, $k \in \{1,2,\ldots,n\}$ and $\mathcal{P}, \mathcal{N}, \mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}}, \mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}}$ as described in Theorems 4 and 5. Then the matrix pencil*

$$sA_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{N}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})} - A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})} \tag{24}$$

is a linearization of the polynomial matrix $T(s)$ using operation free products as coefficients.

Proof It was proved in [1] and [2] that the matrix pencil $sA_{\mathcal{N}} - A_P$ is a linearization of the polynomial matrix $T(s)$. Using strict equivalence and the fact that T_0, T_n are nonsingular, we can conclude that the matrix pencil $A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}})}(sA_{\mathcal{N}} - A_P)A_{(\mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})}$ is also a linearization of the polynomial matrix $T(s)$. We now need to show that its coefficients are operation free products.

- In view of Lemma 9 $(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{N}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})$ is equivalent to $(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}}, \mathcal{L}_{\mathcal{N}}, \mathcal{N}, \mathcal{R}_{\mathcal{N}})$. From Theorem 4, $(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}})$ satisfies the SIP and so does $(\mathcal{L}_{\mathcal{N}}, \mathcal{N}, \mathcal{R}_{\mathcal{N}})$. Writing $A_{(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}}, \mathcal{L}_{\mathcal{N}}, \mathcal{N}, \mathcal{R}_{\mathcal{N}})} = A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}})}A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{N}, \mathcal{R}_{\mathcal{N}})}$ and noting that $A_{(\mathcal{L}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}})} = diag\{X, I_{n-k+1}\}$ and $A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{N}, \mathcal{R}_{\mathcal{N}})} = diag\{I_{k-1}, Y\}$ (Lemmas 10 and 11), we can conclude that $A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{N}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})}$ is operation free.

– In view of Lemma 9 $(\mathcal{L}_\mathcal{N}, \mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P}, \mathcal{R}_\mathcal{N})$ is equivalent to $(\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N}, \mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P})$. From Theorem 5, $(\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N})$ satisfies the SIP and so does $(\mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P})$. Writing $A_{(\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N}, \mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P})} = A_{(\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N})} A_{(\mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P})}$ and noting that $A_{(\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N})} = diag\{Y, I_k\}$ and $A_{(\mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P})} = diag\{I_{n-k}, X\}$ (Lemmas 10 and 11), we can conclude that $A_{(\mathcal{L}_\mathcal{N}, \mathcal{L}_\mathcal{P}, \mathcal{P}, \mathcal{R}_\mathcal{P}, \mathcal{R}_\mathcal{N})}$ is operation free.

Theorem 6 is the main theorem of this paper, describing a new larger family of linearizations. The constraint of T_0, T_n nonsingular can be relaxed using the following remark.

Remark 2 The constraint of T_0 being nonsingular is needed only in the case that in one (or even in both) $\mathcal{L}_\mathcal{P}, \mathcal{R}_\mathcal{P}$ exists the 0 index. The constraint of T_n being nonsingular is needed only in the case that in one (or even in both) $\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N}$ exists the index $-n$.

Proof The transforming matrices in the proof of Theorem 6 are $A_{(\mathcal{L}_\mathcal{N}, \mathcal{L}_\mathcal{P})}$ and $A_{(\mathcal{R}_\mathcal{P}, \mathcal{R}_\mathcal{N})}$. If T_0 is singular but $0 \notin \mathcal{L}_\mathcal{P}, \mathcal{R}_\mathcal{P}$ both transforming matrices are nonsingular. If T_n is singular but $-n \notin \mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N}$ both transforming matrices are nonsingular.

Obviously the family of linearizations in Theorem 6 includes linearizations that have not appeared in [1] and [2] since in the products constructing the coefficients, some elementary matrices appear more than once.

An interesting aspect of the family of linearizations 24 is that it includes all symmetric linearizations constructed by the matrices termed S_i in [8], [9] and [10] which has been shown to form a basis of the symmetric linearizations vector space in $\mathbb{L}_1 \cap \mathbb{L}_2$ in [11]. Also note that in the construction of S_i both T_0 and T_n should be nonsingular.

Corollary 1 *Let $T(s)$ a regular polynomial matrix of degree n, with T_0, T_n nonsingular. Then the matrices S_k in in [8], [9] and [10] are constructed as*

$$S_k = A_{(\mathcal{I}_k, \mathcal{I}'_k)} = A_{(\mathcal{I}'_k, \mathcal{I}_k)}, \ k = 0, \ldots, n$$

where

$$\begin{aligned} \mathcal{I}_k &= ((0 : k-1), \ldots, (0 : 0)) \\ \mathcal{I}'_k &= ((-n : -k-1), \ldots, (-n : -n)). \end{aligned}$$

Proof The proof is constructive noting that $A_{\mathcal{I}_1} = diag\{X_k, I_{n-k}\}$ and $A_{\mathcal{I}_2} = diag\{I_k, Y_{n-k}\}$ where X_k and Y_{n-k} are matrices of block dimensions $k \times k$ and $n-k \times n-k$ respectively.

The matrix pencils $L_k(s) = sS_{k-1} - S_k$, $k = 1, \ldots, n$ have been proven to be linearizations of the polynomial matrix $T(s)$. In the following we will show that they are included in our new family of linearizations.

Corollary 2 *The symmetric linearizations $L_k(s) = sS_{k-1} - S_k, k = 1, \ldots, n$ of [8], [9] and [10] are produced by 24 by setting in Theorem 6*

$$\begin{aligned} \mathcal{P} &= (0 : k-1), \\ \mathcal{N} &= (-n : -k), \\ \mathcal{R}_\mathcal{P} &= ((0 : k-2), \ldots, (0 : 0)), \\ \mathcal{R}_\mathcal{N} &= ((-n : -k-1), \ldots, (-n : -n)), \\ \mathcal{L}_\mathcal{P} &= \varnothing, \mathcal{L}_\mathcal{N} = \varnothing. \end{aligned}$$

Proof Since $\mathcal{I}_k = (\mathcal{P}, \mathcal{R}_{\mathcal{P}})$, $\mathcal{I}_k^{'} = \mathcal{R}_{\mathcal{N}}$, $\mathcal{I}_{k-1} = (\mathcal{R}_{\mathcal{P}})$, $\mathcal{I}_{k-1}^{'} = (\mathcal{N}, \mathcal{R}_{\mathcal{N}})$ the matrix pencil $sS_{k-1} - S_k = sA_{(\mathcal{I}_{k-1}, \mathcal{I}_{k-1}^{'})} - A_{(\mathcal{I}_k, \mathcal{I}_k^{'})}$ can be written as $sA_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{N}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})} - A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})}$ and so using Theorem 6 we conclude that is a linearization included in the family 24.

Example 7 (Symmetric linearizations of multiplicative approach) Let $T(s)$ a $p \times p$ polynomial matrix of degree $n = 5$ with T_0 and T_n nonsingular. Then setting $k = 1$ in the previous corollary we have

$$
\begin{aligned}
\mathcal{P} &= (0), \\
\mathcal{N} &= (-5, -4, -3, -2, -1), \\
\mathcal{R}_{\mathcal{N}} &= (-5, -4, -3, -2, -5, -4, -3, -5, -4, -5), \\
\mathcal{L}_{\mathcal{P}}, \mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{P}} &= \varnothing
\end{aligned}
$$

and

$$
L_1(s) = sS_0 - S_1 = sA_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{N}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})} - A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})} =
$$

$$
sA_{(\mathcal{N}, \mathcal{R}_{\mathcal{N}})} - A_{(\mathcal{P}, \mathcal{R}_{\mathcal{N}})} = s \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 \\ T_2 & T_3 & T_4 & T_5 & 0 \\ T_3 & T_4 & T_5 & 0 & 0 \\ T_4 & T_5 & 0 & 0 & 0 \\ T_5 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -T_0 & 0 & 0 & 0 & 0 \\ 0 & T_2 & T_3 & T_4 & T_5 \\ 0 & T_3 & T_4 & T_5 & 0 \\ 0 & T_4 & T_5 & 0 & 0 \\ 0 & T_5 & 0 & 0 & 0 \end{bmatrix}.
$$

Notice that one can easily check that $L_1(s)$ is a block symmetric linearizations using 11, since $\overline{(\mathcal{N}, \mathcal{R}_{\mathcal{N}})} \sim (\mathcal{N}, \mathcal{R}_{\mathcal{N}})$ and $\overline{(\mathcal{P}, \mathcal{R}_{\mathcal{N}})} \sim (\mathcal{P}, \mathcal{R}_{\mathcal{N}})$. Correspondingly by using $k = 3$ we have

$$
\begin{aligned}
\mathcal{P} &= (0, 1, 2), \\
\mathcal{N} &= (-5, -4, -3), \\
\mathcal{R}_{\mathcal{P}} &= (0, 1, 0), \\
\mathcal{R}_{\mathcal{N}} &= (-5, -4, -5), \\
\mathcal{L}_{\mathcal{P}}, \mathcal{L}_{\mathcal{N}} &= \varnothing
\end{aligned}
$$

and

$$
L_3(s) = s \begin{bmatrix} 0 & -T_0 & 0 & 0 & 0 \\ -T_0 & -T_1 & 0 & 0 & 0 \\ 0 & 0 & T_3 & T_4 & T_5 \\ 0 & 0 & T_4 & T_5 & 0 \\ 0 & 0 & T_5 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -T_0 & 0 & 0 \\ 0 & -T_0 & -T_1 & 0 & 0 \\ -T_0 & -T_1 & -T_2 & 0 & 0 \\ 0 & 0 & 0 & T_4 & T_5 \\ 0 & 0 & 0 & T_5 & 0 \end{bmatrix}.
$$

Similarly we can construct all the other linearizations appearing in [8], [9] and [10].

Although rewriting the matrices S_i and the corresponding linearizations using products of elementary matrices is by itself an important theoretical tool providing in our opinion more insight and a more natural method for their construction, it will be shown in the following that it allows more control, producing several block symmetric linearizations of similar form that have not appeared in the literature before, where additionally the constraint of T_0 and T_n being nonsingular can be relaxed. This can be seen in the following example.

Example 8 (Symmetric linearizations without nonsingularity constraints) Let $T(s)$ a $p \times p$ polynomial matrix of degree $n = 5$ **without** the constraint of nonsingularity for T_0 and T_n. Then by setting $k = 1$ and

$$\begin{aligned} \mathcal{P} &= (0), \\ \mathcal{N} &= (-1, -3, -2, -5, -4), \\ \mathcal{R}_{\mathcal{N}} &= (-3), \\ \mathcal{L}_{\mathcal{N}} &= (-3, -2, -4), \\ \mathcal{L}_{\mathcal{P}} &= \mathcal{R}_{\mathcal{P}} = \varnothing, \end{aligned}$$

we have

$$\begin{aligned} L_1^{'}(s) &= sA_{(\mathcal{L}_{\mathcal{N}},\mathcal{L}_{\mathcal{P}},\mathcal{N},\mathcal{R}_{\mathcal{P}},\mathcal{R}_{\mathcal{N}})} - A_{(\mathcal{L}_{\mathcal{N}},\mathcal{L}_{\mathcal{P}},\mathcal{P},\mathcal{R}_{\mathcal{P}},\mathcal{R}_{\mathcal{N}})} \\ &= s\begin{bmatrix} T_1 & T_2 & T_3 & I & 0 \\ T_2 & T_3 & T_4 & 0 & I \\ T_3 & T_4 & T_5 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -T_0 & 0 & 0 & 0 & 0 \\ 0 & T_2 & T_3 & I & 0 \\ 0 & T_3 & T_4 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix} \end{aligned}$$

is a new symmetric linearization of $T(s)$. Equivalently setting $k = 3$ and

$$\begin{aligned} \mathcal{P} &= (1, 2, 0), \\ \mathcal{N} &= (-5, -3, -4), \\ \mathcal{R}_{\mathcal{P}} &= (1), \\ \mathcal{L}_{\mathcal{N}} &= (-4), \\ \mathcal{L}_{\mathcal{P}} &= \mathcal{R}_{\mathcal{N}} = \varnothing, \end{aligned}$$

we have that

$$L_3^{'}(s) = s\begin{bmatrix} 0 & I & 0 & 0 & 0 \\ I & -T_1 & 0 & 0 & 0 \\ 0 & 0 & T_3 & T_4 & I \\ 0 & 0 & T_4 & T_5 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ 0 & -T_0 & -T_1 & 0 & 0 \\ I & -T_1 & -T_2 & 0 & 0 \\ 0 & 0 & 0 & T_4 & I \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

is a new symmetric linearization of $T(s)$. Using similar arguments we can construct $S_4^{'}$ as

$$L_5^{'}(s) = s\begin{bmatrix} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ I & 0 & -T_1 & -T_2 & 0 \\ 0 & I & -T_2 & -T_3 & 0 \\ 0 & 0 & 0 & 0 & T_5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & -T_0 & -T_1 & -T_2 \\ I & 0 & -T_1 & -T_2 & -T_3 \\ 0 & I & -T_2 & -T_3 & -T_4 \end{bmatrix}.$$

Notice that in the light of Remark 2, $L_1^{'}(s)$, $L_3^{'}(s)$ and $L_5^{'}(s)$ are linearizations even if T_0 and T_n are singular.

Obviously all the linearizations in the previous examples do not belong in the class of linearizations proposed in [1] and [2]. Also the linearizations $L_i^{'}(s)$ do not belong in the family of linearizations $\mathbb{L}_1$, $\mathbb{L}_2$ introduced in [11], a fact that can be confirmed using the shifted sum characterization of $\mathbb{L}_1$, $\mathbb{L}_2$ used therein.

As a brief example on the simple construction of symmetric linearizations consider a regular polynomial matrix $T(s)$ of degree 4 with T_0 non-singular and the pencil $s\tilde{E}-\tilde{A}$ with $\tilde{E} = A_2 A_0 A_{-4}$ and $\tilde{A} = A_2 (A_3 A_0 A_1 A_2) A_0$. Clearly $s\tilde{E}-\tilde{A}$ is a linearization of $T(s)$. Then $\tilde{E}$ is block symmetric since $(2,0,-4) \sim \overline{(2,0,-4)}$ and so is $\tilde{A}$ since $(2,3,0,1,2,0) \sim \overline{(2,3,0,1,2,0)}$. So $s\tilde{E}-\tilde{A}$ is a block symmetric pencil of the form

$$s\begin{bmatrix} -T_0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & -T_2 & 0 \\ 0 & 0 & 0 & T_4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -T_0 & 0 \\ 0 & 0 & 0 & I \\ -T_0 & 0 & -T_1 & -T_2 \\ 0 & I & -T_2 & -T_3 \end{bmatrix}.$$

Note also that this linearization also does not belong in the family of block symmetric linearizations $\mathbb{L}_1 \cap \mathbb{L}_2$ in [11].

4 Conclusions

In this paper we have established a new family of linearizations of polynomial matrices. The linearizations are constructed using products of elementary matrices and have a companion like structure. Some of the linearizations have been found to be block symmetric and a subset of them do not require the nonsingularity constraints of T_0 and T_n that usually appear in the literature. An interesting topic for further research, is to apply these new linearizations in the case of $T(s)$ being block skew symmetric, palindromic etc.

References

1. E.N. Antoniou, S. Vologiannidis, *A new family of companion forms of polynomial matrices*, Electronic Journal of Linear Algebra, 2004, Vol. 11, pp 78-87
2. E.N. Antoniou, S. Vologiannidis, *Linearizations of polynomial matrices with symmetries and their applications*, Electronic Journal of Linear Algebra, volume 15 (2006), pages 107-114.
3. Miroslav Fiedler. *A note on companion matrices*, (2003) Linear Algebra and Its Applications, 371, pp. 325-331.
4. Miroslav Fiedler,. *Intrinsic products and factorizations of matrices,* Linear algebra and its applications, 2008, vol. 428, no1, pp. 5-13.
5. Fernando De Teran, Frolian Dopico and D. Steven Mackey, *Linearizations of Singular Matrix Polynomials and the Recovery of Minimal Indices*, March 2009, MIMS EPrint: 2009.25, ISSN 1749-9097, http://www.manchester.ac.uk/mims/eprints.
6. Nicholas J. Higham, D. Steven Mackey, Niloufer Mackey, Francoise Tisseur, *Symmetric Linearizations for Matrix Polynomials,* SIAM Journal on Matrix Analysis and Applications, Volume 29 , Issue 1 (December 2006), pp 143-159
7. N. P. Karampetakis, *Matrix Pencil Equivalents of Symmetric Polynomial Matrices*, Special Issue: Recent Developments in Multidimensional Systems, Control and Signals—Theory and Applications, Asian Journal of Control, Vol.12, No.2, 2010.
8. P. Lancaster, *Symmetric transformations of the companion matrix*, NABLA: Bulletin of the Malayan Math. Soc., 8 (1961), pp. 146–148.
9. P. Lancaster and U. Prells, *Isospectral families of high-order systems*, ZAMM · Z. Angew. Math. Mech. 87, No. 3, 219 – 234 (2007).
10. P. Lancaster, *Lambda-Matrices and Vibrating Systems*, Pergamon Press, Oxford, UK, 1966. Reprinted by Dover, New York, 2002.
11. D. Steven Mackey, Niloufer Mackey, Christian Mehl and Volker Mehrmann, *Vector Spaces of Linearizations for Matrix Polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971-1004.

12. D. Steven Mackey, Niloufer Mackey, Christian Mehl and Volker Mehrmann, *Structured polynomial eigenvalue problems: Good vibrations from good linearizations*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1029-1051.
13. Vardulakis, A.I.G., *Linear Multivariable Control - Algebraic Analysis and Synthesis Methods*, John Willey & Sons Ltd, New York, 1991.